Math 259A Extra Note

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November 25, 2019

1 Student Presentations

In this class, every enrolled student gave a presentation on a topic. Here are notes I took for each presentation.

1.1 Kadison's transitivity theorem

Definition 1.1. If M is a C^*-algebra acting on a Hilbert space H, M is said to act **topologically irreducibly** if H has no proper, closed, invariant subspaces under M. M is said to act **algebraically irreducibly** if H has no proper, invariant subspaces under M.

From the definitions, we have that algebraically irreducible C^*-algebras are topologically irreducible.

Theorem 1.1 (Kadison's transitivity theorem)**.** *If M is topologically irreudicble, it is algebraically irreducible.*

Why is this called the transitivity theorem? We will show that M acts n-transitively on H; i.e. for all linearly independent $x_1, \dots, x_n \in H$ and any $y_1, \dots, y_n \in H$, there is an $A \in M$ such that $Ax_i = y_i$ for all $1 \leq i \leq n$.

Lemma 1.1. *Let $x_1, \dots, x_n \in H$ be orthonormal, and let $z_1, \dots, z_n \in H$ with $\|z_i\| \leq r$. Then there exists an operator $B \in \mathcal{B}(H)$ such that $Bx_i = z_i$ for all i and $\|B\| \leq \sqrt{2n}r$. If there is a selfadjoint T with $Tx_i = z_i$, then we can take B to be self-adjoint.*

Proof. Extend $x_1, \dots, x_n, x_{n+1}, \dots, x_m$ to an orthonormal basis for $\mathbb{C}\{x_1, \dots, x_n, z_1, \dots, z_n\}$ ($m < 2n$). Let $\widetilde{B}$ be the matrix induced by splitting up the z_i according to this basis. Then

$$[\widetilde{B}] = \sqrt{\sum |\alpha_{i,j}|^2} \leq (2n \cdot r^2)^{1/2} = \sqrt{2n}r.$$

Extend it by making it 0 on the orthogonal complement. $\square$

Proof. Assume $x_1, \ldots, x_n$ are orthonormal, so $x_1, \ldots, x_n \xrightarrow{B} y_1, \ldots, y_n$. By changing basis and conjugating by change of basis operators, we can get this result for arbitrary sets. Choose B_0 such that $B_0 x_i = y_i$. Take $A_0 \in M$ such that $\|A_0 x_i = y_i\| \leq \frac{1}{2\sqrt{2n}}$; this is possible because M is topologically irreducible. Choose B_1 such that $B_1 x_i = y_i - A_0 x_i$ and $\|B_1\| \leq \frac{1}{2}$. By Kaplansky's density theorem, choose $A_1 \in M$ such that $\|A_1\| \leq \frac{1}{2}$ and $A_1 x_i - B_1 x_i\| \leq \frac{1}{4\sqrt{2n}}$.

Continue recursively: Suppose we have defined B_k such that $\|B_k\| \leq \frac{1}{2^k}$ and $B_k x_i = y_i - A_0 x_i - A_1 x_i - \cdots - A_{k-1} x_i$. Choose $A_k \in M$ such that $\|A_k\| \leq \frac{1}{2^k}$, $\|A_k x_i - B_k x_i\| \leq \frac{1}{2^{k+1}\sqrt{2n}}$. Choose $\|B_{k+1}\| \leq \frac{1}{2^{k+1}}$ with $B_{k+1} x_i = y_i - A_0 x_i - A_1 x_i - \cdots - A_k x_i$. If $T x_i = y_i$, we can choose the B_k and thus the A_k to be self-adjoint by Kaplansky's theorem. Let $A = \sum_{k=0}^{\infty} A_k$, This converges in norm to an element of M. Moreover,

$$y_i - A x_i = y_i - \sum_{k=0}^{\infty} A_k x_i = \lim_k (y_i - a_0 x_i - A_1 x_i - \cdots - A_k x_i) = \lim_k (B_{k+1} x_i) = 0$$

because $\|x_i\| = 1$ and $\|B_{k+1}\| \leq 1/2^{k+1}$. This proves n-transitivity and thus Kadison's theorem. $\square$

1.2 Dixmier's averaging theorem

Theorem 1.2 (Dixmier's averaging theorem)**.** *Let M be a von Neumann algebra with center $Z(M)$. For each $x \in M$, denote by $\overline{K(x)}$ the norm closure of the convex hull of $\{uxu^* : u \in U(M)\}$. Then $\overline{K(x)} \cap Z(M) \neq \varnothing$.*

The bulk of the proof is in the following lemma.

Lemma 1.2. *If $x = x^* \in M$, there is a $u \in U(M)$ and $y = y^* \in Z(M)$ such that*

$$\left\| \frac{1}{2}(x + u^* x u) - y \right\| \leq \frac{3}{4}\|x\|.$$

Proof. Suppose $\|x\| = 1$. Define projections $p = \mathbb{1}_{[0,1]}(x)$ and $q = \mathbb{1}_{[-1,0]}(x)$. By the comparison theorem, there exists some $z \in P(Z(M))$ such that $zq \prec zp$ and $(1-z)p \prec (1-z)q$. Take p_1, p_2, q_1, q_2 such that $zq \sim p_1 \leq p_1 + p_2 = 2p$ and $(1 - zp) \sim q - 1 \leq q_1 + q_2 = (1-z)q$.

Take two partial isometries $v, w \in M$ with $c^* c = w$ and $vv^* = p$, $w^* w = (1-z)p$, $vv^* = q$. Define $u = v + v^* + w + w^* + p_2 + q_2$. Then

$$\begin{aligned} u &= v^* v + vv^* + w^8 w + ww^* + q_2 + p_2 \\ &= zq + p_2 + (1-z)p + q_1 + q_2 + p_2 \\ &= p + q \\ &= 1. \end{aligned}$$

Also,

$$u^*p_1u = zq, \qquad u^*q_1u = (1-z)p \qquad u^*p_2u = p_2,$$
$$u^*zqu = p_1, \qquad u^*(1-z)pu = q_1, \qquad u^*q_2u = q_2.$$

We have $-zq \le zx \le zp = p_1 + p_2$. So

$$\begin{aligned}
&\implies -p_1 \le zu^*xu \le zq + p_2 \\
&\implies -\frac{1}{2}(zq+p_1) \le \frac{1}{2}(zx + zu^*xu) \le \frac{1}{2}zq + p_1 + p_1 \\
&\implies \frac{1}{2}z \le \frac{1}{2}(2x + zu^*xu) \le z \\
&\implies -\frac{3}{4} \le \frac{1}{2}(2x - zu^*xu) - \frac{1}{4}z \le \frac{3}{4}z.
\end{aligned}$$

Similarly, repeating this with $1-z$ gives

$$-\frac{3}{4}(1-z) \le \frac{1}{2}((1-z)x + (1-z)u^*xu) + \frac{1}{4}(1-z) \le \frac{3}{4}(1-z).$$

If we add these together, we get

$$\left\| \frac{1}{2}(z + u^*xu) - \frac{2z-1}{4} \right\| \le \frac{3}{4}. \qquad \square$$

Proof. Let K denote the set of maps $\alpha : M \to M$ of the form $\alpha(x) = \sum_{i=1}^n c_i u_i^* x u_i$ with $u_i \in U(M)$, $\sum_i c_i = 1$ and $c_i \ge 0$. For general $x \in M$ denote $a_0 = \mathrm{Re}(x)$ and $b_0 = \mathrm{Im}(z)$. By the lemma, there exist some $u \in U(M)$ and $y_1 = y_1^* \in Z(M)$ with

$$\left\| \frac{1}{2}(a_0 + u^*a_0u) - y_1 \right\| \le \frac{3}{4}\|a_0\|.$$

Denote $\alpha_1(x) = \frac{1}{2}(x + u^*xu)$ and $a_1 = \alpha_1(a_0)$. Use the lemma again on $a_1 - y_1$. Continue inductively.

Given any $\varepsilon > 0$, we can find $\alpha \in K$ and $y \in Z(M)$ for which $\|\alpha(a_0) - y\| < \varepsilon$. Similarly, given this α, we can find $\beta \in K$ and $z \in Z(M)$ for which $\|\beta(\alpha(b_0)) - z\| < \varepsilon$. Thus,

$$\|\beta(\alpha(a_0)) - y\| = \|\beta(\alpha(a_0) - y)\| \le \|\alpha(a_0) - y\| < \varepsilon.$$

Therefore,

$$\|\beta(\alpha(x)) - (y + iz)\| < 2\varepsilon$$

The problem is that $y + iz$ might be dependent on ε. To fix that, we define a sequence $(\Gamma_n) \subseteq K$ and $(z_n) \subseteq Z(M)$ such that if $x_0 = x$ and $x_n = \gamma_n(x_{n-1})$, we have $\|x_n - z_n\| \le \frac{1}{2^n}$. Thus,

$$\|x_{n+1} - x_n\| = \|\gamma_{n+1}(x_n - z_n) - (x_n - z_n)\| \le \|\gamma_{n+1}(x_n - z_n)\| + \|x_n - z_n\| < \frac{1}{2^{n-1}}.$$

Thus, $x_n \to x$ and $z_n \to x$, so $x \in \overline{K(x)} \cap Z(M)$. $\square$

1.3 The Ryll-Nardzewski fixed point theorem

I gave this presentation. See my notes on the subject.

1.4 $\ell^1(\mathbb{Z})$ is not a C^*-algebra

Theorem 1.3. *$\ell^1(\mathbb{Z})$ is not a C^*-algebra.*

Theorem 1.4. *Let $\varphi \in C(S^1)$ with $\varphi(z) = 0$ for all $z \in S^1$. Then $\widehat{\varphi} \in \ell^\infty(\mathbb{Z})$, $\widehat{x\varphi} \in \ell^\infty(\mathbb{Z})$.*

These are consequences of the following fact.

Theorem 1.5. *Let $\Omega(\ell^1(\mathbb{Z}))$ be the maximal ideal space of $\ell^1(\mathbb{Z})$. $\Omega(\ell^1(\mathbb{Z})) \cong S^1$, where $\Omega(\ell^1(\mathbb{Z}))$ is equipped with the weak topology in $(\ell^1(\mathbb{Z}))^* \cong \ell^\infty(\mathbb{Z})$.*

Proof. Let i denote the natural isomorphism from $(\ell^1(\mathbb{Z}))^* \to \ell^\infty$. We claim that $i(\Omega(\ell^1(\mathbb{Z}))) = \{\alpha \in \ell^\infty(\mathbb{Z}) : \alpha(m+n) = \alpha(m)\alpha(n)\}$.

For any $\varphi \in \Omega(\ell^1(\mathbb{Z}))$ with $i(\varphi) = \alpha\varphi$,

$$\alpha\varphi(m+n) = \sum \delta_{m+n}\alpha_\varphi = \varphi(\delta_{m+n}) = \varphi(\delta_m * \delta_n) = \varphi(\delta_m)\varphi(\delta_n) = \alpha_\varphi(m) \cdot \alpha_\varphi(n).$$

On the other hand, if $\alpha(m+n) = \alpha(m) \cdot \alpha(n)$, then

$$\begin{aligned} i^{-1}(\alpha)(f * g) &= \sum (f * g)\alpha \sum_i \sum_j f(i-j)g(j)\alpha(i) \\ &= \sum_j \sum_i f(i-j)g(j)\alpha(i-j)\alpha(j) \\ &= \langle g, \alpha \rangle \langle f, \alpha \rangle \\ &= i^{-1}(\alpha(f)) \cdot i^{-1}(\alpha(g)). \end{aligned}$$

Now observe that $\alpha(m) = (\alpha(1))^m$, which gives a bijection $\widehat{\mathbb{Z}} \to A^1$ by $\alpha \mapsto \alpha(1)$. These spaces are compact, so we only need to check continuity of the map to get a homeomorphism. If $\alpha_i \xrightarrow{wk} \alpha$, then

$$\alpha_i(1) = \sum \delta_1 \alpha_i \to \sum \delta_1 \alpha = \alpha(1).$$

So we get that $S^1 \cong \widehat{\mathbb{Z}} \cong \Omega(\ell^1(\mathbb{Z}))$. □

Now we can show that $\ell^1(\mathbb{Z})$ is not a C^*-algebra.

Proof. Assume $\ell^1(\mathbb{Z})$ is a C^*-algebra. Then by the Gelfand transform, $\ell^1(\mathbb{Z}) \cong C(S^1)$. Then $\Gamma(\ell^1(\mathbb{Z})) = \{\varphi \in C(S^1) : \widehat{\varphi} \in \ell^1(\mathbb{Z})\}$.

We claim that $\widehat{\Gamma}(f) = f$, where $f \in \ell^1(\mathbb{Z})$. If $\Gamma(f) \in C^1(S^1)$, then $\Gamma(f)(z) = \langle f, z^n \rangle = \sum f(n) z^n$. We check

$$\widehat{\Gamma(f)}(n) = \frac{1}{2\pi} \int_0^{2\pi} \sum f(n) e^{inx}) e^{inx} \, dx = f(n).$$

We now claim that if $\varphi \in C(S^1)$ then $\widehat{\varphi} \in \ell^1(\mathbb{Z})$. We have

$$^\wedge(\Gamma(\widehat{\varphi}) - \varphi) - \widehat{\Gamma(\widehat{\varphi})} - \widehat{\varphi} = 0$$

by the first claim. □

Here is the proof of the other result.

Proof. $\Gamma(f)$ is invertible if and only if f is invertible. Then if $\varphi = \Gamma(f)$, then $1/\varphi = \Gamma(f^{-1})$. □

1.5 There are no nontrivial projections in $C^*_{\text{red}}(\mathbb{F}_2)$.

This is a presentation about Effros' paper, "Why the circle is connected." In a more concrete sense, this is about the fact that the reduced C^*-algebra of $\mathbb{F}_n$ has no nontrivial projections.

To begin, let's motivate and define a connected C^*-algebra by examining $C(X)$ for a compact topological space X. If X is connected and $P \in \text{Proj}(C(X))$, then $P = 0$ or 1. This is because $P^{-1}((1/2, \infty))$ and $P^{-1}((-\infty, 1/2))$ cover X. So we define connectedness for a C^*-algebra as follows.

Definition 1.2. A C^*-algebra M is **connected** if $\text{Proj}(M) = \{I, 0\}$.

Consider $C(S^1)$. By Fourier series, $C(S^1) \cong C^*_{\text{red}}(\mathbb{Z})$, the reduced C^*-algebra. $\mathbb{Z}$ is the free group on 1 generator. This is why this is related to the circle.

Theorem 1.6. *There are no nontrivial projections in* $C^*_{\text{red}}(\mathbb{F}_2)$.

With slight modifications, the argument we will make will generalize, with modifications to $\mathbb{F}_n$. The idea is that we will get two traces on $C^*_{\text{red}}(\mathbb{F}_2)$, one of which (ordinary trace - tr) is always $\mathbb{Z}$-valued on projections, and the other, τ, is faithful and unital. Then we will find τ in terms of tr and use the following lemmas.

Lemma 1.3. *If* τ *is*

1. *faithful* ($\tau(a^*a) \geq 0$ *for all* a *with* $\tau(a^*a) = 0 \implies a = 0$*),*
2. *unital* ($\tau(1) = 1$*),*
3. *tracial* ($\tau(ab) = \tau(ba)$*),*

then $\tau(\mathrm{Proj}(M)) \subseteq \mathbb{Z}$. *So there are no nontrivial projections.*

Proof. Let P be a projection. Since $P^*P = P$, $\tau(P^*P) = \tau(P) \geq 0$ (with equality iff $P = 0$). The same is true for $1 - P$. So $0 \leq P \leq 1$. But $\tau(P) = 0$ or 1. If the former occurs, then since τ is faithful, $P = 0$. If the latter occurs, $\tau(1 - P(= \tau(1) - \tau(P) = 0$, so $1 - P = 0$. So $P = 1$. $\square$

Lemma 1.4. *Let* P, Q *be projections in* $\mathcal{B}(H)$ *and suppose that* $P - Q$ *is trace class (i.e.* $\mathrm{tr}(|P - Q|) < \infty$*, so* $\mathrm{tr}(P - Q)$ *is independent of basis). Then* $\mathrm{tr}(P - Q) \in \mathbb{Z}$.

Here, $\mathrm{tr}(A) = \sum_k \langle e_k, Ae_k \rangle$, where e_k is an orthonormal basis of the space.

Proof. First, note that

$$\begin{aligned} P(P-Q)^2 &= P(P + Q - PQ - QP) \\ &= P + PQ - PQ - PQP \\ &= P - PQP, \end{aligned}$$

$$\begin{aligned} (P-Q)^2 P &= (P + Q - PQ - QP)P \\ &= P - PQP. \end{aligned}$$

So $P(P-Q)^2 = (P-Q)^2P$. Similarly, $Q(P-Q)^2 = (P-Q)^2Q$. So $(P-Q)^2$ is positive, and $\mathrm{tr}((P-Q)) < \infty$, so $(P-Q)^2$ is a Hilbert-Schmidt operator. So $(P-Q)^2$ is compact.

We get from the spectral theorem that

$$(P-Q)^2 = \sum_k \lambda_k P_k,$$

where λ_k are eigenvalues with $\lambda_k > 0$ and P_k are projections. We can take $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$, and we have $\lim_{k\to\infty} \lambda_k = 0$. Define $q = 1 - \sum_k P_k$. Observe that $(P-Q)q = 0$, as

$$\begin{aligned} \langle (P-Q)qx, (P-Q)qx \rangle &= \langle qx, (P-Q)^2 qx \rangle \\ &= \left\langle qx, \sum_k \lambda_k P_k qx \right\rangle \\ &= 0, \end{aligned}$$

where $P_k q = 0$ for all k. Also, $PP_k = P_kP$ and $QP_k = P_kQ$ (as $P_k = f_k((P-Q)^2)$ for some continuous f_k on $\mathrm{Spec}((P-Q)^2)$). So

$$P - Q = \sum_k (P-Q)P_k = \sum_k PP_k - \sum_j QP_j.$$

This gives us that

$$\mathrm{tr}(P-Q) = \sum \mathrm{tr}((P-Q)P_k) = \sum_k \mathrm{tr}(PP_k) - \mathrm{tr}(QP_k).$$

Moreover, $\mathrm{tr}(PP_k)$ and $\mathrm{tr}(QP_k)$ are integers because they are finite dimensional projections (The trace of a finite dimensional projection is its dimension.) □

We can show that a set is connected by showing that any probability measure on the set gives 0 or 1 measure to a clopen set. This will be similar to what we are doing. The trace we will define will be analogous to integrating against Lebesgue measure.

The second trace we are interested in is $\tau_{\mathbb{F}_2}$ given by

$$\tau(a) = \langle e_1, \lambda(a)e_1\rangle$$

where $a \in C^*_{\mathrm{red}}(\mathbb{F}_2)$, and λ is left multiplication.

Remark 1.1. Suppose $a = \sum a_g \ell_g$. Then $\langle e_h, a_g e_{gh}\rangle = a_g \delta_{h,gh}$, so $g = 1 = a_1$. Such a trace is faithful and unital (done in class).

Compare this to $\mathrm{tr}(a)$: The trace tr is a sum of terms like

$$\langle e_g, \lambda(a)e_g\rangle = \langle e_1, \lambda(a)e_1\rangle = \tau(a).$$

We have $\lambda : C^*_{\mathrm{red}}(\mathbb{F}_2) \to \mathcal{B}(\ell^2(\mathbb{F}_2))$. We will define $\lambda_0 : C^*_{\mathrm{red}}(\mathbb{F}_2) \to \mathcal{B}(\ell^2(\mathbb{F}_2))$ as follows. Write $\mathbb{F}_2 = S_u \cup S_v \cup \{e\}$, where S_u is the set of words that end with u or u^{-1} and S_v is the set of words that end with v or v^{-1}. This gives $\ell^2(\mathbb{F}_2) = H_u \oplus H_V \oplus \mathbb{C}e_1$.

H_u and H_v are isomorphic to $\mathbb{F}_2$ in the sense that $\mathbb{F}_2$ acts in the same way on them. What is the action/representation? Define

$$\lambda_0(u)e_1 = 0, \qquad \lambda_0(u)e_g = \begin{cases} e_{ug} & \text{if } g \neq u^{-1} \text{ or } 1 \\ e_u & g = u^{-1} \end{cases}, \qquad \lambda_0(v)e_g = \begin{cases} e_{vg} & \text{if } g \neq v^{-1} \text{ or } 1 \\ e_u & g = u^{-1}. \end{cases}$$

This defines a representation $\lambda_0 : C^*_{\mathrm{red}}(\mathbb{F}_2) \to \mathcal{B}(\ell^2(\mathbb{F}_2))$ where $\lambda_0(e_1) = 0$; this is the only thing in the kernel. We have $\lambda|_{H_u} = \lambda_0|_{H_u}$ and $\lambda|_{H_v} = \lambda_0|_{H_v}$. So we get

$$\lambda_0 \cong \lambda|_{H_u} \oplus \lambda|_{H_v} \oplus 0_{e_1}.$$

Let's compare λ and λ_0. If $a \in C^*_{\mathrm{red}}(\mathbb{F}_2)$, then $\tau(a) = \langle e_g, \lambda(a)e_g\rangle$. On the other hand, if $s \in S_u$, then

$$\langle e_s, \lambda_0(u)e_s\rangle = \langle e_t, \lambda(u)e_t\rangle = \tau(u)$$

for some t. But $\langle e_1, \lambda_0(u)e_1\rangle = 0$. So the diagonal entries of $\lambda(u)$ and $\lambda_0(u)$ differ only at e_1. By induction on the length of a word, we get that $\lambda(a)$ and $\lambda_0(a)$ differ only at finitely many places for $a \in \mathbb{C}\mathbb{F}_2$.

We want to say that $\tau(a) = \text{tr}(\lambda(a) - \lambda_0(a))$. But we don't know that $\lambda(a) - \lambda_0(a)$ is trace class in general. Define $\mathscr{A}_0 \subseteq C^*_{\text{red}}(\mathbb{F}_2)$ by $a \in A_0$ if and only if $\lambda(a) - \lambda_0(a)$ is trace class. We want to show that if $P \in \text{Proj}(C^*_{\text{red}}(\mathbb{F}_2))$ with $P \neq 0, I$, then there exists an $e \in \text{Proj}(\mathscr{A}_0)$ with $e \neq 0, I$. But then $\tau(e) = \text{tr}(\lambda(e) - \lambda_0(e)) \in \mathbb{Z}$, which is a contradiction by the first lemma.

If $a \in \mathbb{C}\mathbb{F}_2$, $\lambda(a)$ and $\lambda_0(a)$ differ at only finitely many places, so $\lambda(a) - \lambda_0(a)$ is finite rank and is hence trace class. So $\mathbb{C}\mathbb{F}_2 \subseteq \mathscr{A}_0$. Since $\mathbb{C}\mathbb{F}_2$ is dense in $C^*_{\text{red}}(\mathbb{F}_2)$, we have an $a \in \mathbb{C}\mathbb{F}_2$ s.t. $\|a - P\| < 1/3$; we can choose a to be self-adjoint. Then $\text{Spec}(a)$ is contained in the union of some neighborhood in $\mathbb{R}$ about 0 and some neighborhood in $\mathbb{R}$ about 1.

We can't just use continuous functional calculus, because we might get something out of it which is not trace class. Instead, define e by

$$e = \frac{1}{2\pi i} \int_\Gamma (z - a)^{-1} \, dz,$$

where Γ is a closed contour around the part of $\text{Spec}(a)$ near 1. Since $\lambda(a)$ and $\lambda_0(a)$ differ at only finitely many places, the same is true for $\lambda(z - a)$ and $\lambda_0(z - a)$ (and we have a uniform bound on the dimension of $\ker(\lambda - \lambda_0)^\perp$). Let R_n be the n-th approximating Riemann sum. Then $R_n \to e$ in the operator norm topology, so $|R_n| \to |e|$ in the operator norm. Also, $\|R_n\| \leq C$ for all n. Moreover, if A is finite rank, $\text{tr}(|A|) \leq \|A\| \cdot \dim(\text{im}(A))$, which gives us a uniform bound on $\text{tr}(|\lambda((z - a)^{-1}) - \lambda_0((z - a)^{-1})|)$. Then use the fact that if $A_n \to A$ in the operator norm and $\text{tr}(|A|) \leq C$, then $\text{tr}(|A|) \leq C$. Hence, e is trace class, so $\tau(e) = \text{tr}(\lambda(e) - \lambda_0(e) \in \mathbb{Z}$.